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ARTICLE *in* ELECTROMAGNETICS · FEBRUARY 2014

Impact Factor: 0.26 · DOI: 10.1080/02726343.2013.863672

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Electromagnetics, 34:85–91, 2014

ISSN: 0272-6343 print/1532-527X online
DOI: 10.1080/02726343.2013.863672

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Abstract *For the first time, a non-iterative formula is derived for the wave number of the Goubau line with a small cross-section. This formula makes use of Lambert's W function. Its accuracy is assessed with the help of a few examples.*

Keywords Goubau line, wave number, wavenumber, Lambert function, wire, dielectric, coat, sleeve, waveguide, electromagnetics

1. Introduction

The so-called "Goubau line" (Goubau, 1950a) shown in Figure 1 is a metal wire (radius a) with a dielectric coat (radius b and relative permittivity ε_r). Its ability to guide waves has been known for more than a century (Harms, 1907). As presented in Goubau's patent application (Goubau, 1950b), this transmission line is operated under the mode that has axial symmetry and zero cutoff frequency, that is, the TM_0 mode (Harms, 1907; Goubau, 1950a, 1950b; Collin, 1960; Harrington, 1961) with wave number k_z. This mode radiates no energy; it propagates some, as $k_0 < k_z < k_0\sqrt{\varepsilon_r}$ (Harrington, 1961), where k_0 is the wave number of free propagation in vacuo.

The *straight* Goubau line has found application in gas sensors (Xu & Bosisio, 2005) and recently in the monitoring of beam line elements in particle accelerators (Musson et al., 2009). Yet it is no match to the coaxial line, when it comes to guided transmission's immunity in the world of telecommunications. Indeed it is prone to outside interference at such discontinuities as bends etc. (Goubau, 1956). This is why there is a lack of simple formulae for predicting this line's behavior in terms of the design parameters. The literature points at *numerical* methods when it comes to computing k_z.

A formula that is not iterative is derived herein for k_z for the first time. Its accuracy is assessed by means of comparing with exact solutions in several examples. It is assumed hereafter that the wire and its coat are a perfect conductor and a lossless nonmagnetic dielectric, respectively, and that $\theta_0 = k_0 b \ll 1$.

Received 22 July 2013; accepted 15 September 2013.
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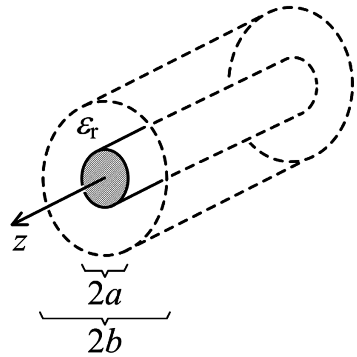

Figure 1. Goubau line.

2. Theory

The characteristic equation of the Goubau line's TM_0 mode is (Harms, 1907; Goubau, 1950a, 1950b; Collin, 1960; Harrington, 1961)

$$\varepsilon_r \theta_{\rho 0} \cdot K_0(\theta_{\rho 0}) \cdot (J_0(\alpha\theta_{p\varepsilon}) \cdot Y_1(\theta_{\rho\varepsilon}) - J_1(\theta_{\rho\varepsilon}) \cdot Y_0(\alpha\theta_{\rho\varepsilon}))$$
$$= \theta_{\rho\varepsilon} \cdot K_1(\theta_{\rho 0}) \cdot (J_0(\theta_{\rho\varepsilon}) \cdot Y_0(\alpha\theta_{\rho\varepsilon}) - J_0(\alpha\theta_{\rho\varepsilon}) \cdot Y_0(\theta_{\rho\varepsilon})), \quad (1)$$

where

$$\begin{cases} \theta_{\rho 0} = \sqrt{\theta_z^2 - \theta_0^2} \\ \theta_{\rho\varepsilon} = \sqrt{\varepsilon_r \theta_0^2 - \theta_z^2} \end{cases}, \quad (2)$$

$\theta_z = k_z b$ and $\alpha = a/b$. J_n and Y_n are the nth-order Bessel functions of the first kind and second kind, respectively. K_n is the second kind modified Bessel function of order n. Equation (1) will be solved for $\theta_{\rho 0} \ll 1$ ($\theta_{\rho 0} = 0$ if $\theta_0 = 0$); then θ_z will obtained from Eq. (2).

Let B be J_n or Y_n; Goubau simplified Eq. (1) by replacing $B(\alpha\theta_{\rho\varepsilon})$ with

$$B(\alpha\theta_{\rho\varepsilon}) \cong B(\theta_{\rho\varepsilon}) + (\alpha - 1)\theta_{\rho\varepsilon}\left(\frac{d}{dx}B(x)\right)_{x=\theta_{\rho\varepsilon}} \quad (3)$$

in the case of a thin coat ($b - a \ll a$) (Goubau, 1950a, 1950b; Collin, 1960). For the following derivations to apply also in the case of a thick coat, one uses

$$B(\alpha\theta_{\rho\varepsilon}) \cong B(\theta_{\rho\varepsilon}) + \sum_{n=1}^{3}(\alpha - 1)^n \frac{\theta_{\rho\varepsilon}^n}{n!}\left(\frac{d^n}{dx^n}B(x)\right)_{x=\theta_{\rho\varepsilon}} \quad (4)$$

instead of Eq. (3). Equation (1) thus becomes

$$\varepsilon_r \theta_{\rho 0} K_0 \left(6 - (1-\alpha)^2(4-\alpha)\theta_{\rho\varepsilon}^2\right) \cong (1-\alpha)(6 + (1-\alpha)(5-2\alpha))\theta_{\rho\varepsilon}^2 K_1, \quad (5)$$

where from Eq. (2),

$$\theta_{\rho\varepsilon}^2 = (\varepsilon_r - 1)\theta_0^2 - \theta_{\rho 0}^2 \quad (6)$$

and (Abramovitz & Stegun, 1970)

$$\begin{cases} K_0 \cong -\gamma - \ln(\theta_{\rho 0}/2), \\ K_1 \cong 1/\theta_{\rho 0} + (\gamma + \ln(\theta_{\rho 0}/2))\theta_{\rho 0}/2 \end{cases} \tag{7}$$

(Euler's constant $\gamma = 0.5772157$). The convenience of Goubau's step (in Eqs. (3) and (4)) proceeds from the absence of *all* the J_n and Y_n terms from Eq. (5). Let $x = \theta_0$, $\alpha - 1$, or $\varepsilon_r - 1$; differentiating Eqs. (1) or (5) twice shows that

$$\begin{cases} \partial\theta_{\rho 0}/\partial x \to 0 \\ \partial^2\theta_{\rho 0}/\partial x^2 \to +\infty \end{cases} \quad \text{when } x \to 0. \tag{8}$$

An attempt to approximate $\theta_{\rho 0}$ by means of a truncated Taylor series, such as

$$\theta_{\rho 0} \cong x\left(\frac{\partial\theta_{\rho 0}}{\partial x}\right)_{x=0} + \frac{x^2}{2}\left(\frac{\partial^2\theta_{\rho 0}}{\partial x^2}\right)_{x=0}, \tag{9}$$

thus fails because of the logarithmic behavior of K_0 from Eq. (7), near $\theta_0 = 0$; Eq. (9) is *not* valid.

Nevertheless, this logarithmic behavior will provide for an approximation of $\ln(\theta_{\rho 0})$. Taking Eqs. (5) through (7) near the limit $\theta_{\rho 0} = 0$ yields

$$\theta_{\rho 0}^2 \cdot \ln\left(\theta_{\rho 0}^2\right) \cong \left(\frac{1}{\varepsilon_r} - 1\right)(1-\alpha)\left(2 + (1-\alpha)\frac{5-2\alpha}{3}\right)\theta_0^2 = \theta_\alpha. \tag{10}$$

For the purpose of solving the general equation,

$$x \cdot \ln(x) = \theta_\alpha, \tag{11}$$

for a given $-1 \ll \theta_\alpha < 0$, Sommerfeld (1899) and Stratton (1941) took advantage of the low rate of the variation of $\ln(x)$ versus x. They obtained x as the limit of a series x_n defined by $x_{n+1} \cdot \ln(x_n) = \theta_\alpha$. But this is an *algorithm* whose rate of convergence depends on starting value x_0. The field analysis that leads to Eq. (1) gives no clue what a good value is for x_0 (Harms, 1907; Goubau, 1950a, 1950b; Collin, 1960; Harrington, 1961). The goal of this study is the derivation of a formula that is *not* iterative. To this effect, one resorts to the product logarithm, also called "Lambert's W function" (Corless et al., 1996), defined as the branches of the inverse of function $x(W) = W \cdot e^W$. Lambert's function is little known among the engineers, for it is not listed in the popular tables of special functions (Abramovitz & Stegun, 1970; Dwight, 1961; Gradshteyn et al., 2000). Some disapproved of Gonnet (Corless et al., 1996) for the same reason, in fact, when this person included this function in Maple™ (Scott et al., 2006).

Rewriting Eq. (10) as

$$\ln\left(\theta_{\rho 0}^2\right) \cdot e^{\ln(\theta_{\rho 0}^2)} \cong \theta_\alpha \tag{12}$$

leads to

$$\theta_{\rho 0} \cong \theta_e = \sqrt{e^{W_{-1}(\theta_\alpha)}}, \tag{13}$$

where W_{-1} is a negative branch of the product logarithm (see Figure 2):

$$\begin{cases} W_{-1}(\theta_\alpha) \cong L_1 - L_2 + L_2/L_2, \\ L_1 = \ln(-\theta_\alpha), \qquad L_2 = \ln(-L_1) \end{cases} \tag{14}$$

with the restriction

$$\theta_\alpha > -e^{-1}. \tag{15}$$

Coming back to Eqs. (5) and (7), Eq. (13) is used to approximate $\ln(\theta_{\rho 0})$:

$$\ln(\theta_{\rho 0}) \cong \ln(\theta_e) = W_{-1}(\theta_\alpha)/2, \tag{16}$$

by virtue of the aforementioned rate of variation of the logarithm. Using Eqs. (6), (7), and (16) in Eq. (5) and neglecting the $\theta_{\rho 0}^4$ and $\theta_0^2\theta_{\rho 0}^2$ terms yields

$$\theta_{\rho 0}^2 \cong \frac{(\varepsilon_r - 1)\theta_0^2}{1 + 6\varepsilon_r K_e/\alpha'}, \tag{17}$$

where $\alpha' = (1-\alpha)(6 + (1-\alpha)(5-2\alpha))$ and $\alpha = a/b$; from Eqs. (7) and (16), $K_e = \ln(2) - \gamma - W_{-1}(\theta_\alpha)/2$ and $\gamma = 0.5772157$; and from Eq. (10), $\theta_\alpha = (1/\varepsilon_r - 1)\alpha'\theta_0^2/3$ and $\theta_0 = k_0 b$. Finally, $k_z = \theta_z/b$ is obtained from Eqs. (2) and (17):

$$\frac{k_z}{k_0} = \frac{\theta_z}{\theta_0} \cong \sqrt{1 + \frac{\varepsilon_r - 1}{1 + 6\varepsilon_r K_e/\alpha'}}, \tag{18}$$

where, according to Eq. (15),

$$\theta_0 < e^{-1/2}/\sqrt{(1 - 1/\varepsilon_r)\alpha'/3} = \theta_{\max}. \tag{19}$$

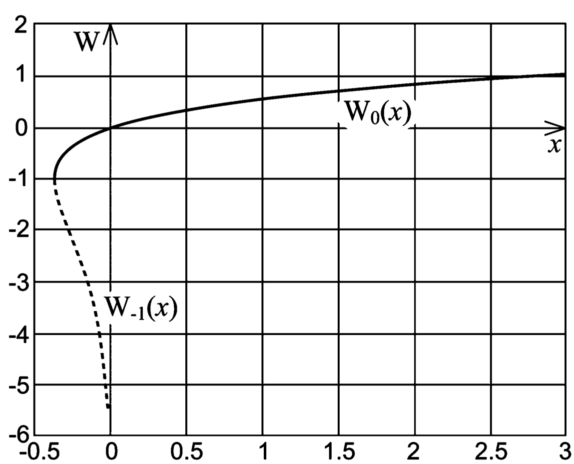

Figure 2. Logarithm product's main two branches.

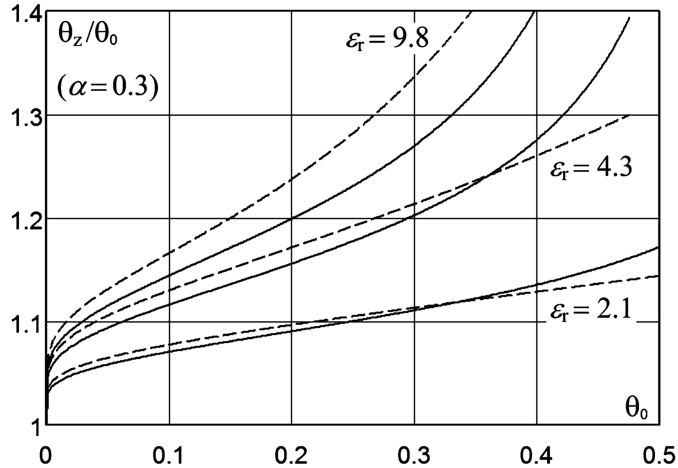

Figure 3. θ_Z/θ_0 from Eqs. (1) and (2) (solid line) and from Eq. (17) (dashed line) for $\alpha = 0.3$.

Equation (19) sets no *physical* limits to Eq. (1) in the sense that the TM_0 mode would *not* propagate if one had $\theta_0 \geq \theta_{max}$. Rather, it states that, let alone accuracy, it makes no sense to extrapolate Eq. (5) along Eq. (10), so to speak, beyond $\theta_0 = \theta_{max}$, from $\theta_0 = 0$. It will be shown in the next section whether Eq. (19) is restrictive in practice, as far as applying Eq. (18). Approximate as it is, the latter equation complies with Eq. (8).

3. Examples

A few examples will help ascertain the range of validity of the approximation in Eq. (18). Figures 3 through 5 show ratio θ_Z/θ_0 as obtained from Eq. (18) and from exact Eq. (1), for $\alpha = 0.3$, 0.5, and 0.9 and for $\varepsilon_r = 2.1$ (teflon), 4.3 (polyamide), and 9.8 (alumina). It can be noted that

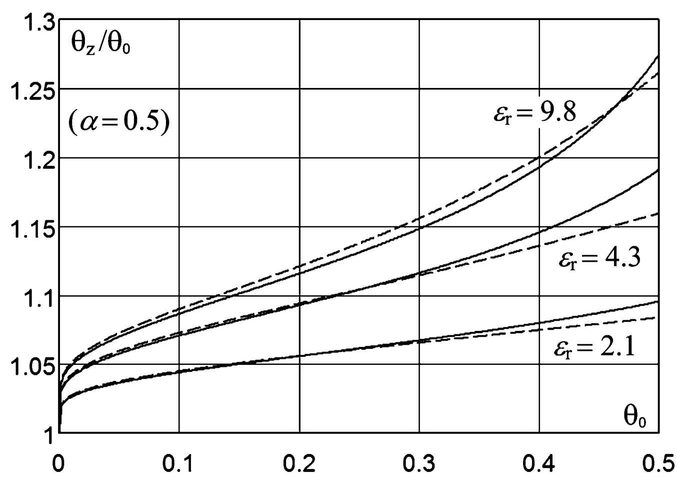

Figure 4. θ_Z/θ_0 from Eqs. (1) and (2) (solid line) and from Eq. (17) (dashed line) for $\alpha = 0.5$.

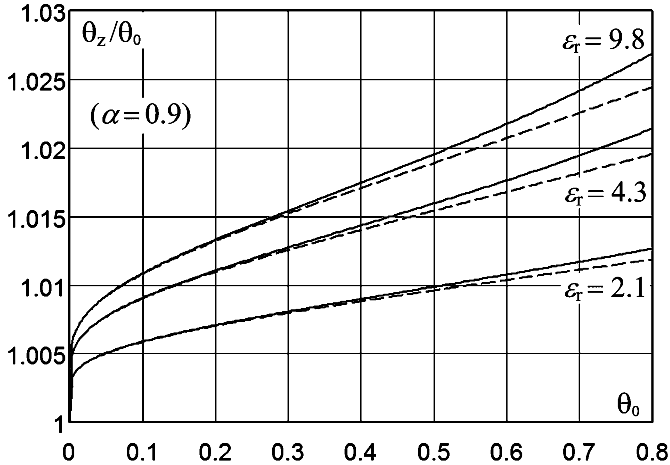

Figure 5. θ_Z/θ_0 from Eqs. (1) and (2) (solid line) and from Eq. (17) (dashed line) for $\alpha = 0.9$.

1. Eq. (18) is usable for the above values of ε_r within ranges $0.5 \leq \alpha \leq 1$ and $0 \leq \theta_0 \leq 0.4$,
2. the above limit $\theta_0 \leq 0.4$ exceeds the starting assumption $\theta_0 \ll 1$ by a factor of 4 or so,
3. Eq. (18) may be used for even smaller values of α, when ε_r is not much greater than unity (see Figure 3);
4. in all cases, the limit set by Eq. (19) is beyond $\theta_0 = 0.4$.

On a more speculative note, it can also be noted that

1. the range of θ_0 would be reduced considerably if the order of the Taylor series for J_n and Y_n were decreased in Eq. (4);
2. this reduction would be even greater if $\theta_{\rho 0} \cong \theta_e$ from Eq. (13) were used up front instead of taking $\theta_{\rho 0}$ from Eq. (17) and $\ln(\theta_{\rho 0})$ from Eq. (16);
3. on the other hand, increasing the order of said series would bring about no improvement that would justify the added complexity;
4. the zeroth-order approximation in Eq. (16) works better in Eq. (17) than the combined first-order

$$\ln(\theta_{\rho 0}) \cong W_{-1}(\theta_\alpha)/2 + (\theta_{\rho 0} - \theta_e)/\theta_e \tag{20}$$

and neglecting the $\theta_{\rho 0}^3$ term generated by Eq. (20).

4. Conclusion

A simple formula has been derived that predicts Goubau line's wave number k_z to engineering accuracy. Its simplicity stems partly from the removal of Bessel functions J_n and Y_n from the TM_0 mode's characteristic equation. The logarithm in K_0 that makes it impossible to write a Taylor approximation for $\theta_{\rho 0}$ (ultimately for k_z) also contributes to its simplicity.

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